MILNOR-WOOD INEQUALITIES FOR PRODUCTS

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Abstract. We prove Milnor-wood inequalities for local products of manifolds. As a consequence, we establish the generalized Chern Conjecture for products $M \times \Sigma^k$ for any product of a manifold M with a product of k copies of a surface Σ for k sufficiently large.

1. Introduction

Let M be an n-dimensional topological manifold. Consider the Euler class $\varepsilon_n(\xi) \in H^n(M, \mathbb{R})$ and Euler number $\chi(\xi) = \langle \varepsilon_n(\xi), [M] \rangle$ of oriented $\mathbb{R}^n$-vector bundles over M. We say that the manifold M satisfies a Milnor-Wood inequality with constant c if for every *flat* oriented $\mathbb{R}^n$-vector bundles ξ over M, the inequality

$$|\chi(\xi)| \leq c \cdot |\chi(M)|$$

holds. Recall that a bundle is flat if it is induced by a representation of the fundamental group $\pi_1(M)$. We denote by $MW(M) \in \mathbb{R} \cup \{+\infty\}$ the smallest such constant.

If X is a simply connected Riemannian manifold, we denote by $\widetilde{MW}(X) \in \mathbb{R} \cup \{+\infty\}$ the supremum of the values of $MW(M)$ when M runs over all closed quotients of X.

Milnor's seminal inequality [Mi58] amounts to showing that the Milnor-Wood constant of the hyperbolic plane $\mathcal{H}$ is $\widetilde{MW}(\mathcal{H}) = 1/2$, and in [BuGe11], we showed that $\widetilde{MW}(\mathcal{H}^n) = 1/2^n$.

We prove a product formula for the Milnor-Wood inequality valid for any closed manifolds:

Theorem 1. *For any pair of compact manifolds M_1, M_2*

$$MW(M_1 \times M_2) = MW(M_1) \cdot MW(M_2).$$

For the product formula for universal Milnor-Wood constant, we restrict to Hadamard manifolds:

Theorem 2. *Let X_1, X_2 be Hadamard manifolds. Then*

$$\widetilde{MW}(X_1 \times X_2) = \widetilde{MW}(X_1)\widetilde{MW}(X_2).$$

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One important application of Milnor-Wood inequalities is to make progress on the generalized Chern Conjecture.

Conjecture 3 (Generalized Chern Conjecture)**.** *Let M be a closed oriented aspherical manifold. If the tangent bundle TM of M admits a flat structure then $\chi(M) = 0$.*

As the name indicates, this conjecture implies the classical Chern conjecture for affine manifolds predicting the vanishing of the Euler characteristic of affine manifolds. This is because an affine structure on M induces a flat structure on the tangent bundle TM.

As pointed out in [Mi58], if $MW(M) < 1$ then the Generalized Chern Conjecture holds for M. Indeed, if $\chi(M) \neq 0$ the inequality

$$|\chi(M)| = |\chi(TM)| \leq MW(M) \cdot |\chi(M)| < |\chi(M)|$$

leads to a contradiction.

One can use Theorem 1 to extend the family of manifolds satisfying the Generalized Chern Conjecture. For instance:

Corollary 4. *Let M be a manifold with $MW(M) < +\infty$. Then the product $M \times \Sigma^k$, where Σ is a surface of genus ≥ 2 and $k > log_2(MW(X))$ satisfies the Generalized Chern Conjecture. In particular, if $\chi(M) \neq 0$, then $M \times \Sigma^k$ does not admit an affine structure.*

Remark 5. *1. One can replace Σ^k in Corollary 4 by any $\mathcal{H}^k$-manifold.*

2. The corollary is somehow dual to a question of Yves Benoist [Be00, Section 3, p. 19] *asking wether for every closed manifold M there exists m such that $M \times S^m$ admits an affine structure. For example, for any hyperbolic manifold M, the product $M \times S^1$ admits an affine structure, but in general $m = 1$ is not enough. Indeed, Sp(2,1) has no nontrivial* 9*-dimensional representations and the dimension of the associated symmetric space is* 8*.*

Note that since there are only finitely many isomorphism classes of oriented $\mathbb{R}^n$-bundles which admit a flat structure, it is immediate that the set

$$\{|\chi(\xi)| \mid \xi \text{ is a flat oriented } \mathbb{R}^n\text{−bundle over } M\}$$

is finite for every M. In particular, if $\chi(M) \neq 0$, there exists a finite Milnor-Wood constant $MW(M) < +\infty$.

However, in general, the Milnor-Wood constant can be infinite. Indeed, the implication $\chi(M) = 0 \Rightarrow \chi(\xi) = 0$, for ξ a flat oriented $\mathbb{R}^n$-bundle, does not hold in general. In Section 5 we exhibit a flat bundle ξ with $\chi(\xi) \neq 0$ over a manifold M with $\chi(M) = 0$. This example is inspired by Smillie's counterexample of the Generalized

Chern Conjecture [Sm77] for nonaspherical manifolds, and likewise this manifold is nonaspherical.

The following questions are quite natural:

(1) Does there exist a finite constant $c(n)$ depending on n only such that $MW(M) \leq c(n)$ for every closed aspherical n-manifold?
(2) Let X be a contractible Riemannian manifold such that there exists a closed X-manifold M with $MW(M) < \infty$. Is $\widetilde{MW}(X)$ necessarily finite?
(3) Does $\chi(M) = 0 \Rightarrow \chi(\xi) = 0$ for flat t oriented $\mathbb{R}^n$-bundles ξ over aspherical manifolds M?

2. Representations of products

Lemma 6. *Let H_1, H_2 be groups and $\rho : H_1 \times H_2 \to GL_n(\mathbb{R})$ a representation of the direct product and suppose that $\rho(H_i)$ is nonamenable for both $i = 1, 2$. Then, up to replacing the H_i's by finite index subgroups, either*

- *$V = \mathbb{R}^n$ decomposes as an invariant direct sum $V = V' \oplus V''$ where the restriction $\rho|V' = \rho'_1 \otimes \rho'_2$ is a nontrivial tensor representation, or*
- *$V = V_1 \oplus V_2$ where G_i is scalar on V_i.*

Proof. This can be easily deduced from the proof of [BuGe11, Proposition 6.1]. □

Proposition 7. *Let $H = \prod_{i=1}^k H_i$ be a direct product of groups and let $\rho : H \to GL_n^+(\mathbb{R})$ be an orientable representation, where $n = \sum_{i=1}^k m_i$. Suppose that $\rho(H_i)$ is nonamenable for every i. Then, up to replacing the H_i's by finite index subgroups $H' = \prod_{i=1}^k H'_i$, either*

(1) *there exists $1 \leq i_0 < k$ such that $V = \mathbb{R}^n$ decomposes nontrivially to an invariant direct sum $V = V' \oplus V''$ and the restricted representation $\rho\,|_{(H'_{i_0} \times \prod_{i>i_0} H'_i, V')}$*
$$H'_{i_0} \times \prod_{i>i_0} H'_i \longrightarrow GL(V')$$
is a nontrivial tensor, or
(2) *the representation ρ' factors through*
$$\rho' : \prod_{i=1}^k H'_i \longrightarrow \left(\prod_{i=1}^k GL_{m'_i}(\mathbb{R})\right)^+ \longrightarrow GL_n^+(\mathbb{R}),$$
where the latter homomorphism is, up to conjugation, the canonical diagonal embedding, and $\rho'(H'_i)$ restricts to a scalar representation on each $GL_{m_j}(\mathbb{R})$, for $i \neq j$.

Moreover, if all m_i are even then either $m_i' < m_i$ for some i or one can replace GL with GL^+ everywhere.

The notation $\left(\prod_{i=1}^k \mathrm{GL}_{m_i'}(\mathbb{R})\right)^+$ stands for the intersection of $\prod_{i=1}^k \mathrm{GL}_{m_i'}(\mathbb{R})$ with the positive determinant matrices.

Proof. We argue by induction on k. For $k = 2$ the alternative is immediate from Lemma 6. Suppose $k > 2$. If Item (1) does not hold, it follows from Lemma 6 that, up to replacing the H_i's by some finite index subgroups, V decomposes invariantly to $V = V_1 \oplus V_1'$ where $\rho(H_1)$ is scalar on V_1' and $\rho(\prod_{i>1} H_i)$ is scalar on V_1. We now apply the induction hypothesis for $\prod_{i>1} H_i$ restricted to V_1'.

Finally, in Case (2), since $\sum m_i = n$, either $m_i' < m_i$ for some i or equality holds everywhere. In the later case, if all the m_i's are even, given $g \in H_i$, since the restriction of $\rho(g)$ each $V_{j\neq i}$ is scalar, it has positive determinant. We deduce that also $\rho(g)|_{V_i}$ has positive determinant. $\square$

3. Multiplicativity of the Milnor-Wood constant for product manifolds – A proof of Theorem 1

Let M_1, M_2 be two arbitrary manifolds. We prove that

$$MW(M_1 \times M_2) = MW(M_1) \cdot MW(M_2).$$

First note that the inequality $MW(M_1 \times M_2) \geq MW(M_1) \cdot MW(M_2)$ is trivial. Indeed, let ξ_1, ξ_2 be flat oriented bundles over M_1 and M_2 respectively of the right dimension such that $|\chi(\xi_i)| = MW(M_i) \cdot |\chi(M_i)|$ for $i = 1, 2$. Then $\xi_1 \times \xi_2$ is a flat bundle over $M_1 \times M_2$ with

$$|\chi(\xi_1 \times \xi_2)| = |\chi(\xi_1)||\chi(\xi_2)| = MW(M_1) \cdot MW(M_2) \cdot |\chi(M_1 \times M_2)|.$$

For the other inequality, let ξ be a flat oriented $\mathbb{R}^n$-bundle over $M_1 \times M_2$, where $n = \mathrm{Dim}(M_1) + \mathrm{Dim}(M_2)$. We need to show that

$$|\chi(\xi)| \leq MW(X_1) \cdot MW(X_2) \cdot |\chi(M)|\,.$$

Observe that if we replace M by a finite cover, and the bundle ξ by its pullback to the cover, then both sides of the previous inequality are multiplied by the degree of the covering.

The flat bundle ξ is induced by a representation

$$\rho : \pi_1(M_1 \times M_2) \cong \pi_1(M_1) \times \pi_1(M_2) \longrightarrow \mathrm{GL}_n^+(\mathbb{R}).$$

If $\rho(\pi_1(M_i))$ is amenable for $i = 1$ or 2, then $\rho^*(\varepsilon_n) = 0$ [BuGe11, Lemma 4.3] and hence $\chi(\xi) = 0$ and there is nothing to prove. Thus, we can without loss of generality suppose that, upon replacing Γ by a finite index subgroup the representation ρ factors as in Proposition 7.

In case (1) of the proposition, we obtain that $\rho^*(\varepsilon_n) = 0$ by Lemma 10 and [BuGe11, Lemma 4.2]. In case (2) we get that ρ factors through

$$\rho : \pi_1(M_1) \times \pi_1(M_2) \longrightarrow \Big(\mathrm{GL}_{m_1'}(\mathbb{R}) \times \mathrm{GL}_{m_2'}(\mathbb{R})\Big)^+ \xrightarrow{i} GL_n^+(\mathbb{R}),$$

where the latter embedding i is up to conjugation the canonical embedding. Furthermore, up to replacing ρ by a representation in the same connected component of

$$\mathrm{Rep}(\pi_1(M_1) \times \pi_1(M_2), \Big(\mathrm{GL}_{m_1'}(\mathbb{R}) \times \mathrm{GL}_{m_2'}(\mathbb{R})\Big)^+)$$

which will have no influence on the pullback of the Euler class, we can without loss of generality suppose that the scalar representations of $\pi_1(M_1)$ on $\mathrm{GL}_{m_2'}$ and $\pi_1(M_2)$ on $\mathrm{GL}_{m_1'}$ are trivial, so that ρ is a product representation. If m_1' or m_2' is odd, then $i^*(\varepsilon_n) = 0 \in H_c^n((\mathrm{GL}_{m_1'}(\mathbb{R}) \times \mathrm{GL}_{m_2'}(\mathbb{R}))^+)$. If m_1' and m_2' are both even then Proposition 7 further tells us that either $m_i' < m_i$ for $i = 1$ or 2, or the image of ρ lies in $\mathrm{GL}^+_{m_1}(\mathbb{R}) \times \mathrm{GL}^+_{m_2}(\mathbb{R})$. In the first case, the Euler class vanishes [BuGe11, Lemma 4.2], while in the second case, we immediately obtain the desired inequality. This finishes the proof of Theorem 1.

4. Multiplicativity of the universal Milnor-Wood constant for Hadamard manifolds - a proof of Theorem 2

Theorem 2 can be reformulated as follows:

Theorem 8. *Let X be a Hadamard manifold with de-Rham decomposition $X = \prod_{i=1}^k X_i$, then $\widetilde{MW}(X) = \prod_{i=1}^k \widetilde{MW}(X_i)$.*

We shall now prove Theorem 8. Note that the inequality "$\geq$" is obvious. Let $M = \Gamma\backslash X$ be a compact X-manifold. We must show that $\mathrm{MW}(M) \leq \prod_{i=1}^k \widetilde{\mathrm{MW}}(X_i)$. Note that Γ is torsion free. Let us also assume that $k \geq 2$. If M is reducible one can argue by induction using Theorem 1. Thus we may assume that M is irreducible. Observe that this implies that $\mathrm{Isom}(X)$ is not discrete. If Γ admits a nontrivial normal abelian subgroup then by the flat torus theorem (see [BH99, Ch. 7]) X admits an Euclidian factor which implies the vanishing of the Euler class. Assuming that this is not the case we apply the Farb–Weinberger theorem [FaWe08, Theorem 1.3] to deduce that X is a symmetric space of non-compact type. Thus, up to replacing M by a finite cover (equivalently, replace Γ by a finite index subgroup), we may assume that Γ lies in $G = \mathrm{Isom}(X)^\circ = \prod_{i=1}^k \mathrm{Isom}(X_i)^\circ = \prod_{i=1}^k G_i$ and G is an adjoint semisimple Lie group without compact factors and $\Gamma \leq G$ is irreducible in the sense that its projection to each factor is dense.

Denote by $\widetilde{G}_i$ the universal cover of G_i, and by $\widetilde{\Gamma} \leq \prod_{i=1}^k \widetilde{G}_i$ the pullback of Γ.

Let $\rho : \Gamma \to \mathrm{GL}_n^+(\mathbb{R})$ be a representation inducing a flat oriented vector bundle ξ over M. Up to replacing Γ by a finite index subgroup, we may suppose that $\rho(\Gamma)$ is Zariski connected. Let $S \leq \mathrm{GL}_n^+(\mathbb{R})$ be the semisimple part of the Zariski closure of $\rho(\Gamma)$, and let $\rho' : \Gamma \to S$ be the quotient representation. By superrigidity, the map $\mathrm{Ad} \circ \rho' : \Gamma \to \mathrm{Ad}(S)$ extends to $\phi : \Gamma \leq \prod_{i=1}^k G_i \to \mathrm{Ad}(S)$ (see [Ma91, Mo06, GKM08]). This map can be pulled to $\widetilde{\phi} : \widetilde{\Gamma} \to S$. Recall also that $\prod_{i=1}^k \widetilde{G}_i$ is a central discrete extension of $\prod_{i=1}^k G_i$ and, likewise, $\widetilde{\Gamma}$ is a central extension of Γ. If $n_i = \dim X_i$ and $n = \sum_{i=1}^k n_i$ we deduce from Proposition 7 and Lemma 10 that either the Euler class vanishes or the image of $\widetilde{\phi}$ lies (up to decomposing the vector space $\mathbb{R}^n$ properly) in $(\prod_{i=1}^k \mathrm{GL}_{n_i})^+$.

Suppose that $\mathrm{MW}(X_i)$ is finite for all $i = 1, \dots, k$ and let M_i be closed X_i-manifolds. Let ξ' be the flat vector bundle on $\prod_{i=1}^k M_i$ coming from $\widetilde{\rho}$ reduced to $\prod_{i=1}^k M_i$, and let ξ_i' be the vector bundle on M_i induced by $\widetilde{\rho}_i$, $i = 1, \dots, k$. By Lemma 9, we have

$$\frac{\chi(\xi)}{\mathrm{vol}(M)} = \frac{\chi(\xi')}{\mathrm{vol}(\prod_{i=1}^k M_i)} = \prod_{i=1}^k \frac{\chi(\xi_i')}{\mathrm{vol}(M_i)} \leq \prod_{i=1}^k \mathrm{MW}(X_i),$$

which finishes the proof of Theorem 8. □

5. Example: a flat bundle with nonzero Euler number over a manifold with zero Euler characteristic

Recall that given two closed manifolds of even dimension, the Euler characteristic of connected sums behaves as

$$\chi(M_1 \sharp M_2) = \chi(M_1) + \chi(M_2) - 2.$$

The idea is to find $M = M_1 \sharp M_2$ such that M_1 admits a flat bundle with nontrivial Euler number in turn inducing such a bundle on the connected sum, and to choose then M_2 in such a way that the Euler characteristic of the connected sum vanishes. Take thus

$$M_1 = \Sigma_2 \times \Sigma_2,\ M_2 = (S^1 \times S^3) \sharp (S^1 \times S^3) \text{ and } M = M_1 \sharp M_2.$$

These manifolds have the following Euler characteristics:

$$\begin{aligned} \chi(M_1) &= 4, \\ \chi(M_2) &= 2\chi(S^1 \times S^3) - 2 = -2, \\ \chi(M) &= 0. \end{aligned}$$

Let η be a flat bundle over Σ_2 with Euler number $\chi(\eta) = 1$. (Note that we know that such a bundle exists by [Mi58].) Let $f : M \to M_1$

be a degree 1 map obtained by sending M_2 to a point, and consider

$$\xi = f^*(\eta \times \eta).$$

Obviously, since η is flat, so is the product $\eta \times \eta$ and its pullback by f. Moreover, the Euler number of ξ is

$$\chi(\xi) = \chi(\eta \times \eta) = 1.$$

Indeed, the Euler number of $\eta \times \eta$ is the index of a generic section of the bundle, which we can choose to be nonzero on $f(M_2)$, so that we can pull it back to a generic section of ξ which will clearly have the same index as the initial section on $\eta \times \eta$.

6. Proportionality principles and vanishing of the Euler class of tensor products

Lemma 9. *Let X be a simply connected Riemannian manifold, $G = Isom(M)$ and $\rho : G \to GL_n^+(\mathbb{R})$ a representation. Then $\frac{\chi(\xi_\rho)}{vol(M)}$, where $M = \Gamma\backslash X$ is a closed X-manifold and ξ_ρ is the flat vector bundle induced on M by ρ restricted to Γ, is a constant independent of M.*

Proof. There is a canonical isomorphism $H_c^*(G) \cong H^*(\Omega^*(X)^G))$ between the continuous cohomology of G and the cohomology of the cocomplex of G-invariant differential forms $\Omega^*(X)^G$ on X equipped with its standard differential. (For G a semisimple Lie group, every G-invariant form is closed, hence one further has $H^*(\Omega^*(X)^G) \cong \Omega^*(X)^G$.) In particular, in top dimension $n = \dim(X)$, the cohomology groups are 1-dimensional $H_c^n(G) \cong H^n(\Omega^*(X)^G)) \cong \mathbb{R}$ and contain the cohomology class given by the volume form ω_X.

Since the bundle ξ_ρ over M is induced by ρ, its Euler class $\varepsilon_n(\xi_\rho)$ is the image of $\varepsilon_n \in H_c^n(\mathrm{GL}^+(\mathbb{R}, n)$ under

$$H_c^n(\mathrm{GL}^+(\mathbb{R}, n) \xrightarrow{\rho^*} H_c^n(G) \longrightarrow H^n(\Gamma) \cong H^n(M),$$

where the middle map is induced by the inclusion $\Gamma \hookrightarrow G$. In particular, $\rho^*(\varepsilon_n) = \lambda \cdot [\omega_X] \in H_c^n(G)$ for some $\lambda \in \mathbb{R}$ independent of M. It follows that $\chi(\xi_\rho)/\mathrm{Vol}(M) = \lambda$. $\square$

Lemma 10. *Let $\rho_\otimes : GL^+(n, \mathbb{R}) \times GL^+(m, \mathbb{R}) \to GL^+(nm, \mathbb{R})$ denote the tensor representation. If $n, m \geq 2$, then*

$$\rho_\otimes^*(\varepsilon_{nm}) = 0 \in H_c^{nm}(GL(n, \mathbb{R}) \times GL(m, \mathbb{R})).$$

Proof. The case $n = m = 2$ was proven in [BuGe11, Lemma 4.1], based on the simple observation that interchanging the two $\mathrm{GL}^+(2, \mathbb{R})$ factors does not change the sign of the top dimensional cohomology class in $H_c^4(\mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R})) \cong \mathbb{R}$, but it changes the orientation on the tensor product, and hence the sign of the Euler class in $H_c^4(\mathrm{GL}^+(4, \mathbb{R}))$.

Let us now suppose that at least one of n, m is strictly greater than 2, or equivalently, that $n + m < nm$. The Euler class is in the image of the natural map

$$H^{nm}(B\mathrm{GL}(nm, \mathbb{R})) \longrightarrow H_c^{nm}(\mathrm{GL}(nm, \mathbb{R})).$$

By naturality, we have a commutative diagram

$$\begin{tikzcd} H^{nm}(B\mathrm{GL}^+(nm, \mathbb{R})) \arrow[r] \arrow[d, "\rho_\otimes^*"] & H_c^{nm}(\mathrm{GL}^+(nm, \mathbb{R})) \arrow[d, "\rho_\otimes^*"] \\ H^{nm}(B(\mathrm{GL}^+(n, \mathbb{R}) \times \mathrm{GL}^+(m, \mathbb{R}))) \arrow[r] & H_c^{nm}(\mathrm{GL}^+(n, \mathbb{R}) \times \mathrm{GL}^+(m, \mathbb{R}))). \end{tikzcd}$$

Since the image of the lower horizontal arrow is contained in degree $\leq n + m$, it follows that $\rho_\otimes^*(\varepsilon_{nm}) = 0$. $\square$

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